Trefftz Approximations: A New Framework for Nonreflecting Boundary Conditions

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A new general framework for approximate nonreflecting boundary conditions in wave scattering involves a set of local Trefftz functions – outgoing waves – and a commensurate set of degrees of freedom (dof). With specific choices of bases and dof, one obtains classical Engquist-Majda and Bayliss-Turkel conditions. Other choices yield a variety of approximate conditions. With additional dof on the artificial boundary, the accuracy of the numerical solution can approach machine precision even on fairly coarse grids, as illustrated numerically.

Index Terms—**Boundary conditions, approximation methods, wave propagation, electrodynamics.**

I. Introduction

THE critical role of artificial boundary conditions or Perfectly Matched Layers for finite difference or finite element solution of wave problems is well known (see e.g. [1]–[6]). We consider the frequency domain wave equation

$$\nabla \cdot \mu^{-1} \nabla u + k_0^2 \epsilon u = f \quad \text{in } \mathbb{R}^n, \quad n = 1, 2, 3; \tag{1}$$

$$\operatorname{supp} f \subset \Omega \subset \mathbb{R}^n, \quad \nu \equiv 1, \ \epsilon \equiv 1 \quad \text{in } \mathbb{R}^n \setminus \Omega$$

In 2D, equation (1) describes the E-mode (TM) if u is a (one-component) electric field, μ and ϵ are the magnetic permeability and dielectric permittivity, respectively; $k_0 = 2\pi/\lambda$ is the vacuum wavenumber. The corresponding equation for the H-mode (TE) is obtained after an obvious change of notation. But the methodology of this paper is general and applicable to more complex cases – 3D and vectorial. As indicated in (1), sources f and scatterers are assumed to be confined to a bounded domain Ω in space.

Eq. (1) is subject to the standard splitting of u into the incident and scattered fields with standard radiation boundary conditions (e.g. Sommerfeld) for the scattered field at infinity. Our task, however, is to replace these theoretical conditions with approximate but accurate and practical ones on the exterior surface $\partial\Omega$ away from the sources and scatterers.

II. The "Trefftz Generator" of Boundary Conditions

The proposed generator of nonreflecting boundary conditions has two main ingredients: (i) a set of local Trefftz functions [8]–[11] ψ_α ($\alpha = 1, 2, \dots, n$) – *outgoing* waves satisfying the wave equation and approximating the solution near a given point on the exterior boundary, and (ii) a set of m degrees of freedom (dof) – linear functionals $l_\beta(u)$ ($\beta = 1, 2, \dots, m$); m is *not* usually equal to n. To elaborate, let the exact solution be approximated locally as a linear combination $u = \sum_\alpha c_\alpha \psi_\alpha = \underline{c}^T \underline{\psi}$, where $\underline{c}$ is a coefficient vector and $\underline{\psi}$ is a vector of basis functions; both vectors are in general complex. Coefficients $\underline{c}$ may be different at different boundary points, but for simplicity of notation this is not explicitly indicated. We are looking for a suitable boundary condition of the form $\sum_\beta s_\beta l_\beta(u) = 0$, where $\underline{s} \in \mathbb{C}^m$ is a set of coefficients ("scheme") to be determined. We require that the scheme be exact for any linear combination of basis functions. Straightforward algebra then yields

$$\underline{s} \in \text{Null } N^T \tag{2}$$

where N^T is an $n \times m$ matrix with entries $N^T_{\alpha\beta} = l_\beta(\psi_\alpha)$. This whole development is completely analogous to that of FLAME [7]–[9], where the dof are the nodal values of the solution on a given grid stencil. It is, however, interesting to consider more general dof – e.g. derivatives. (Gratkowski [12] uses similar ideas to derive analytical boundary conditions for static problems, without the nullspace formula.)

A few interesting combinations of basis functions and dof are summarized in Table I. "Parametric derivatives" of a plane wave mean the following. Let $\psi(x, y, k, \theta) = \exp(-jk(x\cos\theta + y\sin\theta))$ be an outgoing plane wave relative to the half-space $x > 0$. Differentiating this wave successively with respect to k at $k = k_0$ or, alternatively, with respect to θ at $\theta = 0$, one obtains a Trefftz basis set, tailored toward accurate approximation of the solution near normal incidence. (In practice, however, this approximation tends to be good in a broad range of angles.) This basis set, with appropriate derivatives as dof [11], leads to the Engquist-Majda condition.

Furthermore, as also indicated in Table I, radial derivatives $\partial u/\partial r$ at the boundary nodes of the grid can be added to the set of dof. One could view that as "double nodes" carrying two dof at the boundary: solution and its radial derivative. As our numerical results show, these additional dof improve the convergence and accuracy of the method dramatically.

III. Numerical Examples

As illustrative examples, we consider two canonical scattering problems for which analytical solutions are well known: scattering from either a perfectly conducting or a dielectric

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TABLE I

Absorbing Conditions Resulting from Particular Choices of Bases and Dof

Trefftz basis	Dof	Absorbing condition	Properties
Cylindrical harmonics	Solution and its radial derivatives	Bayliss-Turkel	Varying order
Plane wave and its parametric derivatives at normal incidence	Solution and its derivatives	Engquist-Majda	Varying order
A set of outgoing plane waves	Nodal values of the solution on a grid stencil	Trefftz-FLAME absorbing condition	Accuracy reasonable but convergence slow
A set of outgoing plane waves	Nodal values of the solution and its radial derivatives on a grid stencil	Generalized Trefftz-FLAME absorbing condition	Order 6 convergence in 2D for (6+3)-point stencils
A set of outgoing Hankel waves	Nodal values of the solution and its radial derivatives on a grid stencil	Generalized Trefftz-FLAME absorbing condition	Order 6 convergence as above, but much higher accuracy

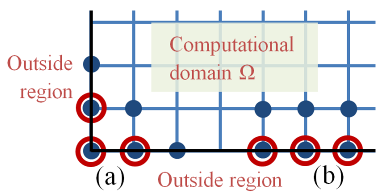

Fig. 1. Grid stencils for nine-point absorbing schemes: (a) at the corner; (b) at the boundary edge. Circles indicate stencil nodes; double circles – "double nodes" carrying two dof, u and $\partial u/\partial r$.

cylinder, infinite in the z direction and with a circular cross-section in the xy-plane. In all cases, a uniform Cartesian grid with size h is introduced. In the bulk, we use 9-point FLAME schemes [8], [9] with either eight plane waves or eight lowest-order cylindrical harmonics as a basis. At the exterior boundary, the scheme is either 6-point or, with three "double nodes" at the boundary, 9-point (Fig. 1). In the latter case, seven outgoing Hankel waves (concentric with the scatterer) are used as a FLAME basis, which produces two (=9–7) independent schemes. The exterior boundary was placed at 1.5λ from the scatterer. Convergence is fundamentally of order 6 (Figs. 2, 3), despite the presence of the air-dielectric interface and the nonreflecting conditions. (Convergence deteriorates slightly when the accuracy approaches machine precision and roundoff errors become noticeable.)

IV. Conclusion

The proposed automatic generator of high-order nonreflecting boundary conditions is based on a set of local Trefftz basis functions (outgoing waves) and a commensurate set of degrees of freedom. The generator reproduces classical Engquist-Majda and Bayliss-Turkel conditions but also opens up avenues for developing new conditions. For canonical problems of 2D scattering, convergence of order six on 9-point stencils is attained, and the relative error of the numerical solution is on the order of 10^{-5} – 10^{-8} with 10 – 20 grid points per vacuum wavelength. We are not aware of alternative methods yielding a similar level of accuracy on comparable grids. Extensions to 3D and vectorial problems are possible.

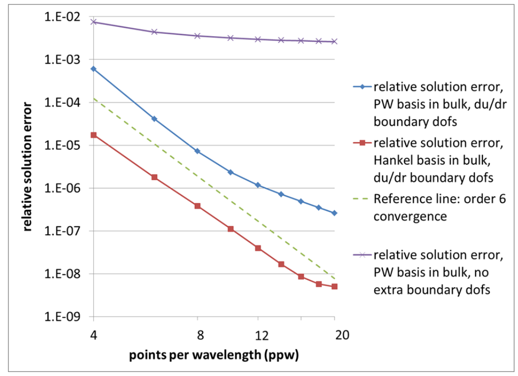

Fig. 2. Relative solution error. Scattering from a perfectly conducting cylinder of radius $r_{\rm cyl} = 0.2\lambda$.

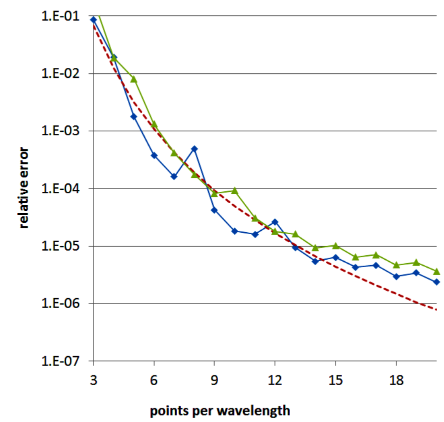

Fig. 3. Relative solution error. Scattering from a dielectric cylinder. $\epsilon_{\rm cyl} = 2$. Diamonds: $r_{\rm cyl} = 0.2\lambda$; triangles: $r_{\rm cyl} = 0.4\lambda$. Dashed line: $O(h^6)$ reference.

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